DGG-003-016201 Seat No. ________

M. Sc. (Sem. II) (Maths) Examination

May/June – 2015

Algebra - II (CMT-2001)

Faculty Code : 003
Subject Code : 016201

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (i) All questions are compulsory.
(ii) Each question carries 14 marks.

1 Choose appropriate alternatives : (any seven) **7×2=14**

(1) Let $g(x) = x^2 - 5x + 6 \in Q[x]$. Which one of following is correct ?

(A) Q contains all the roots of g(x)

(B) g(x) is irreducible over Q

(C) $Q \cap \{\alpha \mid \alpha \text{ is a root of } g(x)\}$ is a single ton set.

(D) None of these

(2) Let $\frac{E}{F}$ and $\frac{K}{E}$ be finite extensions with $[K:E] = [E:F] = 5$.

What is degree of the field extension $\frac{K}{F}$?

(A) 5

(B) 25

(C) 10

(D) None of these

(3) Let F be a subfield of a field E and $\alpha, \beta \in F$. Which one of following is correct ?

(A) $\alpha + \beta \in F$, but $\alpha - \beta \notin F$

(B) $\alpha + \beta, \alpha - \beta \in F$, but $\alpha\beta \notin F$

(C) $\alpha + \beta, \alpha - \beta, \alpha\beta \in F$

(D) $F \cap \{\alpha + \beta, \alpha - \beta, \alpha\beta\} = \phi$

(4) Let F be a finite field. Which one of following is possible ?

(A) $|F| = 91$

(B) $|F| = 810$

(C) $|F| = 15$

(D) $|F| = 9$

(5) Let $f(x) = x^3 + x + 1 \in \mathbb{Z}_2[x]$. Which one of following is correct ?

(A) $f(x)$ is irreducible in $\mathbb{Z}_2[x]$

(B) $\mathbb{Z}_2$ contains a root of f(x)

(C) $\mathbb{Z}_2$ contains all the roots of f(x)

(D) None of these

(6) Let $\alpha = 2^{1/n}$ and $n \geq 3$. Which one of following is correct ?

(A) $\alpha \in Q$

(B) α is not algebraic over Q

(C) $[Q(\alpha) : Q] = 2$

(D) $x^n - 2 \in Q[x]$ is the minimal polynomial of α over Q.

(7) Which of following fields is a prime field ?

(A) Q

(B) $\mathbb{Z}_p$ (p is a prime)

(C) (A) and (B)

(D) None of these

(8) Which of following field extensions is an infinite algebraic extension ?

(A) $\frac{\mathbb{R}}{Q}$

(B) $\frac{\bar{Q}}{Q}$

(C) $\frac{\mathbb{C}}{\mathbb{R}}$

(D) None of these

2 Attempt any two : **2×7=14**

(a) Let $f(x)=a_0+a_1x+..........+a_nx^n \in Z[x]$ and $n>1$. Let p be a prime such that $p|a_0,\ p^2 \nmid a_0,\ p|a_1,.........,p|a_{n-1}$ and $p \nmid a_n$. Then prove that f(x) is irreducible over Q[x].

(b) Let $\frac{E}{F}$ be a finite field extension. Prove that it is also an algebraic extension.

(c) Let $\frac{E}{F}$ and $\frac{K}{E}$ both are algebraic extensions. Then prove that $\frac{K}{F}$ is also an algebraic extension.

(d) Let F be a finite field. Then prove that $F^* = F - \{0\}$ is a cyclic group under multiplication.

3 Attempt any one : **1×14=14**

(a) Let F be a finite field and $|F| = p^n$, for some prime p, $n \in N$. Then prove that Aut(F), the group of all automorphisms of F is a cyclic group of order n.

(b) Let $\frac{E}{F}$, $\frac{K}{E}$ both are finite separable extensions. Prove that $\frac{K}{F}$ is also a finite separable extension.

(c) Let $\frac{E}{F}$ be a Galois extension and K be a subfield of E and it is super field for F. Prove that

(i) $\frac{K}{F}$ is normal $\Leftrightarrow \sigma(K) = K,\ \forall \sigma \in G\left(\frac{E}{F}\right)$

(ii) $\psi = \mathbb{C} \to \mathbb{D}$ defined by $\psi(K) = G\left(\frac{E}{K}\right)$ is bijection, where

$\mathbb{C} = \{K / K \text{ be a field and } F \subseteq K \subseteq E\}$

$\mathbb{D} = \left\{H / H \text{ is a subgroup of } G\left(\frac{E}{F}\right)\right\}$

(iii) $\frac{K}{F}$ is normall $\Leftrightarrow G\left(\frac{E}{K}\right) \Delta G\left(\frac{E}{F}\right)$

(iv) In third case $G\left(\frac{K}{F}\right) \simeq \dfrac{G\left(\frac{E}{F}\right)}{G\left(\frac{E}{K}\right)}$

4 Attempt any two : **2×7=14**

(a) Define prime field. Prove that for a field F, the prime subfield of F is either isomorphic to Q or it is isomorphic with $\mathbb{Z}_p$ (where p is a prime).

(b) Let $f : M \to N$ be an R-homomorphism on R-modules. Prove that Ker f and f(M) are submodules of M and N respectively.

(c) Show that $G\left(\frac{\not\subset}{\mathbb{R}}\right)$ is a group of order 2.

(d) State and prove Hilbert theorem 90.

(e) Let $\frac{E}{F}$ be an extension and $\alpha \in E$ be algebraic over F.

Let $p(x) \in F[x]$ be a polynomial of least degree whose one root is α. Then prove that

(1) p(x) is irreducible over F[x]

(2) p(x)/g(x) in F[x]

5 Attempt any seven : **7×2=14**

(1) Give definition of primitive polynomial and monic polynomial.

(2) Prove that $x^3 + 3x + 2$ is an irreducible polynomial in $\mathbb{Z}_7[x]$.

(3) Define cyclic extension with an example.

(4) Write down all the elements of the splitting field of $x^3 + x^2 + 1$ over $\mathbb{Z}_2$.

(5) Construct a field F with $|F| = 4$.

(6) Define R-module with an example.

(7) Prove or disprove $Q\left(2^{1/3}\right)/Q$ is a normal extension.

(8) Define separable polynomial, separable element and separable extension.
